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2.8. FUZZY EQUIVALENCE OF CLASSICAL CONTROLLERS

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Abstract

This paper concerns the exact equivalence of second order linear discrete controllers. The equivalence principal automatically produces a systematic design procedure for the Fuzzy Controllers. The equivalence of the linear controllers can also be used to implement a nonlinear control behaviour by piecewise linear approximation. Another merit of the equivalence principle is that it provides an opportunity to have a fair comparison between linear and fuzzy controllers.

1. Introduction

Fuzzy logic introduced by Zadeh [1] has found wide applications in the control of industrial systems. However, the classical linear controllers (e.g., PI, PID) are still the most widely used controllers in the practical applications due to their simplicity of design and implementations. This paper addresses the fact that a Fuzzy Controller (FC) for a given input universe of discourse can exactly represent linear discrete controllers. It may be thought that there is no point in implementing a linear control law by a FC; however, this can be a preliminary step in designing FCs for deterministic systems with known non-linearity where the desired nonlinear global behaviour is represented by piecewise linear approximation. In addition, although there are many successful fuzzy speed and position control applications, usually these controllers are designed by trial and error methods [2,3]. The derivation of the fuzzy equivalence of a linear controller generates an automatic design procedure for the FCs. Further, the equivalence principle provides a fair comparison between fuzzy and linear controllers: there are many research papers presenting such performance comparisons between fuzzy and linear controllers [3-6]. It is reasonable to assume that to have a fair comparison, the controllers under evaluation should give exactly or very similar closed loop output responses for same nominal conditions [4]. Hence, when the parameters of the plant are changed or an external disturbance is applied, one can easily see which method gives the more robust control performance. Using the equivalence principle, the FC under evaluation may be designed to satisfy this comparison criteria.

2. Fuzzy Equivalence of a Second Order Linear Discrete Controller

The fuzzy equivalence of a linear discrete PI controller has been considered by Galichet and Foulloy in [7]. In this paper, the fuzzy equivalence of a second order linear discrete controller which has a transfer function

$$G_c(z) = \frac{u(z)}{e(z)} = \frac{K_c(z-a)(z-b)}{(z-1)(z-c)} \qquad (1)$$

will be considered. Note that (1) can be easily converted to a PI, PD or PID controller if the parameters b and c are chosen properly. For example, if c is set to zero then (1) becomes a representation of a PID controller, if b and c are set to zero then it becomes a PI controller. If b is set to 1 and c is set to zero then (1) becomes a representation of a PD controller. Note also that (1) is originally a PI+lead controller (if c < b) and it can also be converted to a lead or lag controller if the parameter a is set to 1, and b and c are chosen according to the desired lead or lag compensation

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In the discrete time domain, the output of the controller (1) can be written as

$$u(k) = u(k-1) + \delta u(k) \tag{2}$$

where

$$\delta u(k) = c\delta u(k-1) + \alpha_1 e(k) + \alpha_2 \delta e(k) + \alpha_3 \delta e(k-1) \tag{3}$$

The constants α_1, α_2 and α_3 are given as

$$\begin{aligned} \alpha_1 &= K_c\left(-(a+b)+ab\right) \\ \alpha_2 &= K_c\left(a+b-ab\right) \\ \alpha_3 &= -abK_c \end{aligned} \tag{4}$$

and the δ operator is defined as

$$\delta x(k) = x(k) - x(k-1) \tag{5}$$

In the following subsections, the fuzzy equivalence of the control law u(k) given by (2) will be considered; however, the output of the FC will be δu(k) rather than u(k) because, in many practical applications, the actuating signal u(k) should be limited (to protect the electronic circuits) with an anti-windup mechanism which stops the integration in the controller. The anti-windup mechanism can be easily implemented in (2) if δu(k) is chosen as the output of the FC (addition ,or integration, is stopped when u(k) reaches the saturation limit, i.e., δu(k) is not added to the previous value u(k-1) during the saturation). Note that u(k) is the numerical integration of δu(k) as seen in (2) which can be easily implemented outside the controller to obtain the actuating signal u(k).

2.1 Sugeno Type Fuzzy Equivalence

In this section, the main purpose is to design a Sugeno type FC that is precisely equivalent to the controller given by (1). A Sugeno type FC has a rule-base consisting of the rules in the form of [8]

IF x_1 is A_1 **AND** x_2 is A_2 **AND**……**AND** x_n is A_n **THEN** $y = g(x_1,x_2,\ldots,x_n)$.

Since the output y is a function of the input variables, any control law can be directly implemented by choosing the output y as the desired control law if the membership functions of the input variables are chosen so that they provide a linear mapping between the inputs and the output of the controller.

As mentioned in Section 2, the output of the FC will be $\delta u(k)$ and u(k) will be obtained by using (2). Thus from (3), the inputs to the FC become $\delta u(k-1)$, e(k), $\delta e(k)$ and $\delta e(k-1)$. In order to keep the FC as simple as possible and to have a linear mapping, the membership functions for the input variables are chosen as shown in Fig.1, where v_i ,for i = 1 to 4, represents the input variables $\delta u(k-1)$, e(k), $\delta e(k)$ and $\delta e(k-1)$ respectively. It should be noted that the input variables are assumed to be bounded and M_i is the maximum value that the magnitude of the corresponding input variable can take on. In other words, M_i determines the limits of the universe of discourse for the corresponding input variable.

The Sugeno type FC will have $2^4 = 16$ rules since there are 4 input variables and 2 membership functions for each input variable. The rules can be represented in a general form as

IF $\delta u(k-1)$ is A_{1m} **AND** e(k) is A_{2n} **AND** $\delta e(k)$ is A_{3p} **AND** $\delta e(k-1)$ is A_{4q} **THEN** $\delta u_{SU}(k) = c\delta u(k-1) + \alpha_1 e(k) + \alpha_2 \delta e(k) + \alpha_3 \delta e(k-1)$

where the indices {m,n,p,q} = {1,2} and α_1, α_2 and α_3 are given by (4).

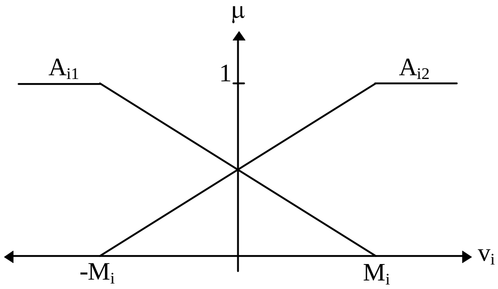

Figure 1 Membership functions for the input variables (i = 1,..,4)

Example 1 : Consider the system shown in Fig.2 :

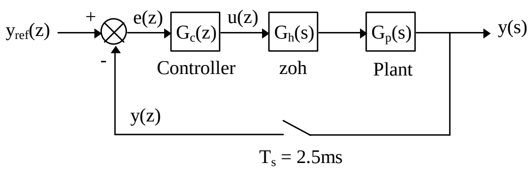

Figure 2 The control system block diagram

The transfer functions of the plant, zero-order-hold (zoh) and the controller are given as

$$G_p(s) = \frac{100}{(s+5)(s+10)} \tag{6}$$

$$G_h(s) = \frac{1-e^{-T_s s}}{s} \tag{7}$$

$$G_c(z) = \frac{2(z-0.9876)(z-0.95)}{(z-1)(z-0.88)} \tag{8}$$

Now the aim is to design a Sugeno type FC that will be precisely equivalent to $G_c(z)$ and thus give exactly the same closed loop responses as the system shown in Fig.2.

The input variables are δu(k-1), e(k), δe(k) and δe(k-1). Their membership functions are chosen as shown in Fig.1, where M_i is selected as 250 for i = 1 to 4 (i.e. for all the input variables). Note that the value of M_i can be chosen arbitrarily high because, as long as the magnitude of the input values do not exceed the corresponding M_i, the equivalence between $G_c(z)$ and the FC will be valid. However, in most of the practical applications, the reference input and the control signal u(k) are usually limited. Hence, all the input variables of the controller are bounded due to these limitations.

The number of rules are 2^4 = 16 (there are 4 input variables and 2 membership functions for each input variable) and the rules can be represented in a general form as

IF δu(k-1) is A_{1m} **AND** e(k) is A_{2n} **AND** δe(k) is A_{3p} **AND** δe(k-1) is A_{4q}**THEN** $\delta u_{SU}(k)$ = 0.88δu(k-1) + 0.00124e(k) + 1.99876δe(k) + 1.87644δe(k-1) where the indices {m,n,p,q} = {1,2}.

Fig.3 shows the simulation results for both $G_c(z)$ and the Sugeno type FC. The output response (y) and the control signal (u) are exactly same for both controllers as expected. The reference input (y_{ref}) is a unit step function applied at t = 0.1 s.

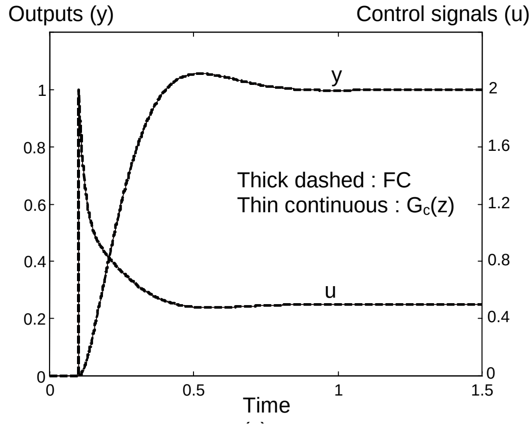

Figure 3 Simulation results showing the equivalence between $G_c(z)$ and the Sugeno type FC

It should be noted that the use of Sugeno type FC becomes more reasonable and meaningful when several control laws are to be implemented in a single controller rather than implementing only one control law. However, in this paper, the main purpose was to illustrate the equivalence between a linear controller and a Sugeno type FC as a preliminary step for the implementation of several control laws in a FC.

2.2 Mamdani Type Fuzzy Equivalence

Mamdani type FCs do not have algebraic equations in the THEN part of the rules [8]; rather they have output membership functions and there is a defuzzification process to produce a control output value. Therefore, the control law of the linear controller can not be directly used in the Mamdani type FCs. Since a linear control law is to be implemented, the inference operators (*AND*, *implication* and *aggregation*) and the defuzzification method should be chosen properly in order to not to lead to a non-linearity in the FC. It is possible to find different ways for implementing a linear control law in a FC, but one of the simplest way is to chose the algebraic product for the *AND* and *implication* operations and to use the center-

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average-defuzzification method [8]. In the center-average-defuzzification method, the aggregation method is not required and the centers of the output membership functions are the quantity of interest, not the shapes of the membership functions. Therefore, the output membership functions can be simply chosen as singletons centred at the appropriate points as shown in Fig.4. It should be noted that the output membership functions do not have to be regularly distributed.

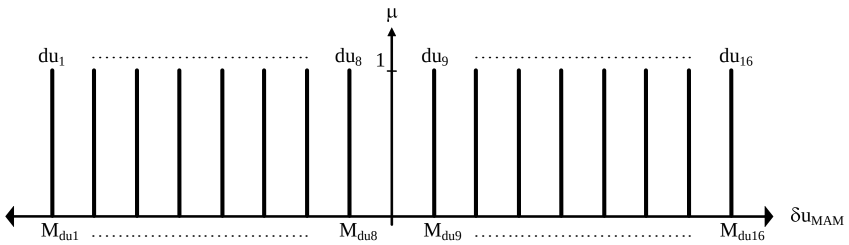

Figure 4 Output membership functions

Table I The rule-base of the Mamdani type FC

$\delta \mathbf{u}_{MAM}$	$\delta u1 = A_{11}$	, $e = A_{21}$
$\delta e1 \setminus \delta e$	A_{31}	A_{32}
A_{41}	$\mathbf{du_1}$	$\mathbf{du_2}$
A_{42}	$\mathbf{du_3}$	$\mathbf{du_4}$

$\delta \mathbf{u}_{MAM}$	$\delta u1 = A_{11}$	, $e = A_{22}$
$\delta e1 \setminus \delta e$	A_{31}	A_{32}
A_{41}	$\mathbf{du_5}$	$\mathbf{du_6}$
A_{42}	$\mathbf{du_7}$	$\mathbf{du_8}$

$\delta \mathbf{u}_{MAM}$	$\delta u1 = A_{12}$	, $e = A_{21}$
$\delta e1 \setminus \delta e$	A_{31}	A_{32}
A_{41}	$\mathbf{du_9}$	$\mathbf{du_{10}}$
A_{42}	$\mathbf{du_{11}}$	$\mathbf{du_{12}}$

$\delta \mathbf{u}_{MAM}$	$\delta u1 = A_{12}$	, $e = A_{22}$
$\delta e1 \setminus \delta e$	A_{31}	A_{32}
A_{41}	$\mathbf{du_{13}}$	$\mathbf{du_{14}}$
A_{42}	$\mathbf{du_{15}}$	$\mathbf{du_{16}}$

The input membership functions should also not introduce any non-linearity. For example, if the input membership functions are chosen as shown in Fig.1, they not only provide a linear mapping but also result in the smallest possible rule-base (i.e. the number of the rules becomes minimum since there are only two membership functions for each input variable).

Using the input and output fuzzy sets shown in Fig.1 and Fig.4, the rule-base of the Mamdani type FC can be given in a tabular form as shown in Table I. The rule-base consists of 16 rules since there are 4 inputs and 2 membership functions for each input variable.

In Table I, the symbols e, δe, $\delta u1$ and $\delta e1$ represent the input variables e(k), $\delta e(k)$, $\delta u(k-1)$ and $\delta e(k-1)$ respectively. The output is represented by δu_{MAM} and the symbols A_{i1}, A_{i2} (i = 1,..,4) and du_1, du_2,……,du_{16} refer to the input and output membership functions shown in Fig.1 and Fig.4 respectively.

Thus, the equivalence problem has been reduced to the determination of the values of M_1,..,M_4 and M_{du1}, M_{du2},…..M_{du16} for the membership functions of the input and output variables respectively. The selection of M_1,..,M_4 has been discussed for the Sugeno type FC in Section 2.1 and this discussion is also valid for the Mamdani type FC since there is no difference between Mamdani and Sugeno type FCs in terms of the input fuzzification process. On the other hand, the output membership function parameters M_{du1}, M_{du2},…..M_{du16} can be determined by using the desired linear control law, the rule-base and the extreme values of the input variables since the FC is expected to implement the desired linear control law between the extremes of the input variables using the rules in the rule-base. For example, from Table I, consider the rule

IF δu1 is A_{11} **AND** e is A_{22} **AND** δe is A_{31} **AND** δe1 is A_{42} **THEN** δu_{MAM} is du_7

which implies that if the certainty of the rule is 1 (that means all the input values are full members of the corresponding fuzzy set, i.e. membership degree = 1, and thus the input values are the extremes), then the output fuzzy set du_7 should have a center at

$$M_{du7} = -cM_1 + \alpha_1 M_2 - \alpha_2 M_3 + \alpha_3 M_4 \qquad (9)$$

to satisfy the equivalence between the controllers at these extreme values of the input variables. In this manner, the parameters M_{du1},....,M_{du16} can be calculated as shown in Table II.

Thus, if the centers of the output membership functions are chosen as shown in Table II, the FC will provide the desired linear control behaviour by implementing a linear interpolation between the output values corresponding to the extremes of the input variables.

Table II The centres of the output membership functions

$M_{du1} = -cM_1 - \alpha_1 M_2 - \alpha_2 M_3 - \alpha_3 M_4$	$M_{du9} = -M_{du8}$
$M_{du2} = -cM_1 - \alpha_1 M_2 + \alpha_2 M_3 - \alpha_3 M_4$	$M_{du10} = -M_{du7}$

$M_{du3} = -cM_1 - \alpha_1 M_2 - \alpha_2 M_3 + \alpha_3 M_4$	$M_{du11} = -M_{du6}$
$M_{du4} = -cM_1 - \alpha_1 M_2 + \alpha_2 M_3 + \alpha_3 M_4$	$M_{du12} = -M_{du5}$
$M_{du5} = -cM_1 + \alpha_1 M_2 - \alpha_2 M_3 - \alpha_3 M_4$	$M_{du13} = -M_{du4}$
$M_{du6} = -cM_1 + \alpha_1 M_2 + \alpha_2 M_3 - \alpha_3 M_4$	$M_{du14} = -M_{du3}$
$M_{du7} = -cM_1 + \alpha_1 M_2 - \alpha_2 M_3 + \alpha_3 M_4$	$M_{du15} = -M_{du2}$
$M_{du8} = -cM_1 + \alpha_1 M_2 + \alpha_2 M_3 + \alpha_3 M_4$	$M_{du16} = -M_{du1}$

Example 2 : Let us again consider the system shown in Fig.2 with the plant and the linear controller given by (6) and (8) respectively. The aim is to design a Mamdani type controller that is precisely equivalent to the linear controller given by (8).

The input membership functions are chosen as shown in Example 1 since there is no difference between Mamdani and Sugeno type controllers in terms of the fuzzification process. Therefore, the membership functions are as shown in Fig.1, where $M_i = 250$ (for i = 1 to 4) for all the input variables (the selection of M_i has been already discussed in Example 1). The output membership functions are chosen as shown in Fig.4 and thus the rule base is as shown in Table I.

By comparing (1) and (8), the linear controller parameters become

$$K_c = 2,\ a = 0.9876,\ b = 0.95 \text{ and } c = 0.88.$$

Using (4), the constants α_1, α_2 and α_3 are calculated as

$$\alpha_1 = 0.00124,\ \alpha_2 = 1.99876 \text{ and } \alpha_3 = -1.87644.$$

The centers of the output membership functions are obtained by using Table II as shown in Table III.

Table III The centres of the output membership functions

M_{du1} = -250.89	M_{du9} = 189.11
M_{du2} = 748.49	M_{du10} = 1188.49
M_{du3} = -1189.11	M_{du11} = -749.11
M_{du4} = -189.73	M_{du12} = 250.27
M_{du5} = -250.27	M_{du13} = 189.73
M_{du6} = 749.11	M_{du14} = 1189.11
M_{du7} = -1188.49	M_{du15} = -748.49
M_{du8} = -189.11	M_{du16} = 250.89

Fig.5 shows the simulation results for the designed Mamdani type FC in comparison with the linear controller (8). The output response (y) and the control signal (u) are exactly same for both controllers as expected. The reference input (y_{ref}) is a unit step function applied at t = 0.1s.

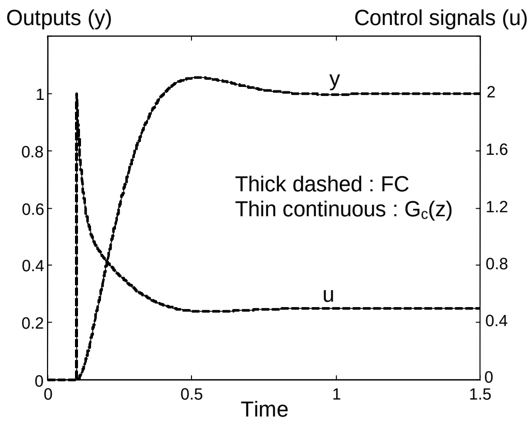

Figure 5 Simulation results showing the equivalence between $G_c(z)$ and the Mamdani type FC

Conclusions

In this paper, the fuzzy equivalence of the second order linear controllers has been investigated. It has been shown that any second order linear control law can be precisely implemented in a FC for a given input universe of discourse. The equivalence may be interpreted as a sort of bridge between the classical and fuzzy control approaches. This is an important point because the equivalence may be used to combine the classical and the fuzzy control approaches in a same framework and thus a controller using the advantages of both control methods may be designed. The fuzzy equivalence of a general controller (n-poles and m-zeros) can also be derived in the similar manner.

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